

27 September 2001
from: Dennis Couzin
to: ASTM Task Group on Flashing Light Measurement
re: Allard 1876

Introduction

Emile Allard in 1876 proposed a remarkably simple formula -- simple to state, not to compute -- for the visibility of flashing light.

Visibility = the maximum value of $i(t)$, where $i(t)$ is the solution of the differential equation

$$\frac{di}{dt} = m \cdot (I(t) - i(t))$$

where $I(t)$ is the intensity of the light at time t , and m is a time constant for the visual system.

This is not intended as a measure of degree of visibility, but for determining whether a detection threshold is met.

Fran oise Vi not kindly supplied the twelve pages pertaining to flashing light from Allard's 1876 publication. The differential equation is reached by the third page. I've studied these first pages, but not the rest, which seem to be mathematical exercises upon the formula. The pages leading to the differential equation are attached below as Appendix A.

Allard arrives at his differential equation *a priori*. He appears to make just one empirical assumption, about exponential decay, from Newton! However his concept of "impression l mentaire" embodies deeper assumptions, and when I reconstruct his mathematically soft argument leading to the equation I find that his assumption about exponential decay is superfluous -- it follows from the deeper assumptions.

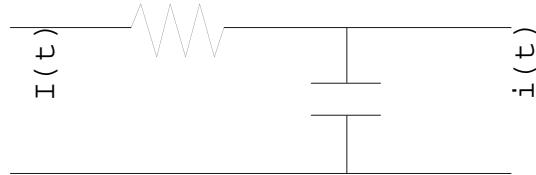
The reconstruction puts Allard's argument on firmer mathematical concepts, making his assumptions more obvious. It is given in the exegesis below.

We will not learn about the visibility of a flash of light directly from Allard's result. The value of his result is that it suggests a more general model: the

"impression" $i(t)$, equals the mathematical convolution of the "lumi re" $I(t)$, with what I have elsewhere called the visual impulse function $q(t)$. By Allard's reasoning, $q(t)$ must equal $m \cdot \exp(-m \cdot t)$ for non-negative t . (Causality requires $q(t) = 0$ for negative t .) I believe the visual impulse function is different from Allard's $q(t)$; it does not reach its maximum value at $t = 0$.

Possibly, with a well-chosen $q(t)$, the visibility of a flash of light can be calculated from the maximum value of a new $i(t)$, the mathematical convolution of $I(t)$ with $q(t)$. Possibly not. When we have good psycho-physical data for a variety of flash forms, or better, a good visual model, will we know.

Allard's original formula allowed a simple instrumental measurement of flash visibility (although not in his time, and this was not his reasoning). If a voltage proportional to light $I(t)$ is applied to an simple R-C circuit then a voltage proportional to impression $i(t)$ is



measured after the capacitor. Indeed Allard's $m = 1/RC$. If we are very very lucky, there is a new circuit producing the new $i(t)$ from $I(t)$.

Exegesis of Allard 1876

p. 62 Linearity. What Allard calls "impressions" are, for constant lights, proportional to intensities.

p. 63 Exponential decay of impressions. When a constant light ends abruptly at time $t=0$, the impression, originally equal to I , obeys the function:

$$i(t) = I e^{-mt} \quad (1)$$

Allard cites analogy with Isaac Newton's considerations on the cooling of small bodies (and no more).

p. 63 Perceptibility based on instantaneous impression.
 If j gives the least perceptible impression, then setting making $i(t) = j$ in equation (1) gives $q = \log_e(I/j)/m$. q measures the duration of the perception.

p. 63 Idea of the elementary impression. An abruptly ending light decays exponentially.

$$di/dt = -m I e^{-mt} = -m i(t) \quad (2)$$

So in the first dt of decay

$$di = -m I dt \quad (3)$$

If having the light off during dt results in an $mI dt$ loss in impression, then for any dt while the light is on, there must be $mI dt$ gained in impression. $mI dt$ is the "impression lem ntaire" produced by the light I during the time dt .

At the abrupt end of a steady light I the falloff follows function $f(t)$. So the falloff in the first dt equals

$$I f'(0)dt \quad (4)$$

(A) So during a steady light I , the light's "contribution" to impression i from t to $t+dt$ must equal $-I f'(0)dt$.

Allard next reasons on the onset of impression for a suddenly started light. The light goes to I . The impression starts at 0, [how do we know this?] and rises as function $i(t)$. Consider the time interval t_1 to t_1+dt .

(B) For the impression to just remain steady at level $i(t)$ should require a contribution of $-i(t_1) f'(0)dt$.

(C) However, the light I makes a contribution of $-I f'(0)dt$ to impression during the interval

(D) Therefore the net rise in impression during the time interval t_1 to t_1+dt is given by

$$di = -I f'(0)dt + i(t_1) f'(0)dt$$

$$di/dt = -f'(0)(I - i) \quad (5)$$

Aside from the strange use of contributions, the particular weakness in the argument is at step (B). Step (B) appropriates from step (A) although in (A), $i(t)$ was steady while in (B) $i(t)$ is rising. There is a hidden assumption that what an impression does from t_1 to t_1+dt can be determined from $i(t_1)$ together with $I(t_1)$ to $I(t_1+dt)$, i.e., that the past history of i need not be given.

Allard is presuming for decay

$$f(t) = e^{-mt} \quad (6)$$

so that $f'(0) = -m$, and

$$di/dt = m(I - i) \quad (7)$$

(Allard's argument would seem to work as well for any other falloff function f , but we will see shortly why this isn't so.)

I now have an improved reconstruction of Allard's argument for equation (7) which does not require his assumption of (6) for decay. Indeed it implies (6).

Allard's notion of "impression lem ntaire" amounts to a contribution by the moment's light to the moment's impression. Since the whole impression is greater than that contribution we must wonder where it comes from. It must come from the past. Suppose a simple conservative model¹. Light I turns on at $t=0$, and follows function $I(t)$. From $t=0$ to arbitrary time t , the light gives

$$\int_{u=0}^{u=t} I(u)du$$

¹A simple model has photons flowing into the eye and electrons flowing out the nerves. This can be conserving or not. Simplest if it is conserving, with delays (= storage) by the eye. Since (7) leads to a convolution model it must be conserving.

toward impression. If the impression follows function $i(t)$ then during the same time, it uses

$$\int_{u=0}^{u=t} i(u) du$$

from what I gave. The remainder

$$\int_{u=0}^{u=t} (I(u) - i(u)) du$$

is stored for the future.

Now suppose the light is abruptly turned off at time t_0 . Allard's prime assumption is that the decay curve for the impression follows some function $f(t)$, regardless of the light's and impression's history. That is, if at t_0 the impression has value $i(t_0)$, then after t_0 it must follow

$$i(t_0)f(t-t_0)$$

Conservation requires that the stored impression equals the integrated impression through the decay.

$$\begin{aligned} \int_{u=0}^{u=t_0} (I(u) - i(u)) du &= \int_{u=t_0}^{\infty} i(t_0)f(u - t_0) du \\ &= i(t_0) \int_{u=t_0}^{\infty} f(u - t_0) du \\ &= i(t_0) \int_{u=0}^{\infty} f(u) du \end{aligned}$$

Whatever f may be its integral from 0 to ∞ , equals some number K . So

$$\int_{u=0}^{u=t_0} (I(u) - i(u)) du = K i(t_0)$$

t_0 could be any time t , so we have arrived at the integral version of Allard's differential equation (7).

$$\int_{u=0}^{u=t} (I(u) - i(u))du = K i(t) \quad (8)$$

$$I(t) - i(t) = K di/dt \quad (9)$$

The number $K = 1/m$ from equation (7). Moreover, if equation (9) is now applied to a constant flash of impression = 1 which ends suddenly at t_f , we get

$$-i(t)dt = K di/dt \quad \text{for } t > t_f$$

$$\text{So} \quad i(t) = i(t_f)e^{-t/K} \quad \text{for } t > t_f$$

This is Allard's equation (6) with $K=1/m$.

The reconstruction gives Allard's "impression lem ntaire" a conservative mechanistic interpretation and then using no more than his prime assumption from step (B) shows how he does not assume exponential decay of impressions, but deduces this.

Allard's equation (7) or (9) imply that impressions must begin to diminish as soon as a light is turned off. This precludes visual delay. Indeed there are plausible visual impulse functions which exhibit such increase.

Allard's prime assumption (B) must be denied.

Appendix A

from "Mémoire sur l'Intensité et la Portée des Phares",
M.E. Allard, Paris, 1876

[p. 62]

tude Théorique de la vision des Feux Scintillants.

L'effet que les feux scintillants produisent sur l'organe de la vue peut donner lieu une tude intéressante, dans laquelle intervient le phénomène connu sous le nom de persistance des impressions sur la rétine. On pourrait croire, au premier abord, qu'en faisant tourner assez rapidement un appareil de feu scintillant, l'impression produite par un des clats, persistant pendant $\frac{1}{10}$ de seconde, viendrait se confondre avec le commencement de l'impression due au clat suivant, de sorte qu'on prouverait la sensation d'un feu fixe beaucoup plus intense que celui qu'on pourrait obtenir de la même lampe avec l'appareil ordinaire. Une proposition dans ce sens avait été faite à l'Administration et a été expérimentée au Département des phares. Le résultat annoncé ne fut pas obtenu, et on reconnut que l'intensité du feu fixe dont la rotation rapide de clats donnait la sensation avait peine à l'intensité de celui qu'on produit avec un appareil immobile.

Nous allons chercher d'expliquer ce qui se passe dans un cas semblable, en étudiant théoriquement la question.

Lorsque l'œil est soumis à l'action d'une source lumineuse d'intensité constante I , la partie de la rétine qu'atteignent les rayons manit de cette source prouve une impression dont l'intensité est proportionnelle à celle de la lumière et dont la mesure peut être prise égale à I . Si la source lumineuse vient à s'éteindre subitement, l'impression sur la rétine ne cesse pas de suite, mais elle diminue suivant une certaine loi et devient bientôt assez faible pour ne plus être perceptible. La loi de cette diminution n'est pas connue, mais il est naturel de supposer qu'elle est analogue à celle que Newton a indiquée pour le refroidissement d'un corps [p. 63] de petites dimensions. On peut donc admettre que la vitesse avec laquelle l'impression diminue est à chaque instant proportionnelle à la grandeur de cette impression, ce qui conduit à l'équation

$$\frac{di}{dt} = -mi, \quad d'où \quad i = I e^{-mt},$$

I tant l'impression au moment où la source lumineuse s'an antit, c'est- -dire pour $t = 0$, i l'impression après le temps t , e la base des logarithmes n p riens et m une constante.

Si on appelle j la plus petite impression perceptible, la valeur de θ tirée de l'équation

$$j=Ie^{-m\theta}$$

sera la durée de la sensation lumineuse que suit la disparition de la lumière. Cette durée variera avec I , mais la variation sera d'autant moindre que m sera plus grand. Au-delà de la durée θ , les valeurs de l'impression ne deviennent jamais théoriquement nulles, mais elles sont extrêmement petites et ne produisent aucune sensation.

L'équation précédente donne, pour $t=0$,

$$di=-mIdt;$$

c'est la perte que l'impression prouve dans le premier instant dt qui suit l'extinction de la lumière, et comme l'impression que fait prouver une lumière constante est elle-même uniforme, cette quantité $mIdt$ doit aussi représenter l'impression mentaire que la lumière I produit pendant chaque instant dt . Cette remarque va nous servir dans la question suivante.

Supposons maintenant qu'une lumière I commence subitement à frapper la rétine. L'impression n'atteindra pas tout de suite sa valeur définitive I , mais elle y parviendra en augmentant à partir de zéro suivant une certaine loi qu'il est facile de déterminer. Soit en effet i la valeur de l'impression au bout du temps t . Supposons qu'au ce moment la lumière s'anantisse: l'impression diminuera suivant la loi précédemment admise, et, dans l'instant dt qui suivra l'extinction de la lumière, la perte sera $-midt$. Mais si la source lumineuse, au lieu de disparaître, continue à agir, elle ajoutera pendant ce même instant dt une impression mentaire [p. 64] qui, comme nous l'avons remarqué ci-dessus, aura pour valeur $mIdt$; de sorte que l'accroissement effectif de l'impression sera

$$di = m(I-i)dt, \quad d'où \quad i=I(1-e^{-mt}).$$